Econ 802
Answers to Second Midterm

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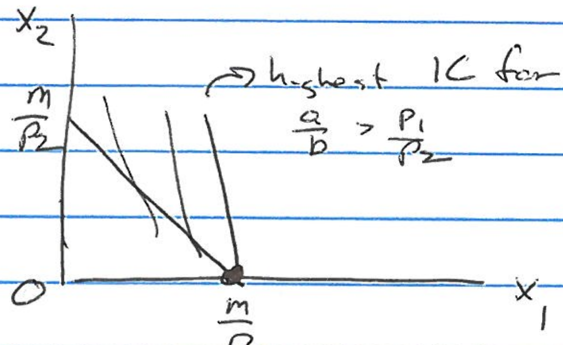
1(a) Fix a budget line using (p,m). The indifference curves are linear with slope $= -\frac{a}{b}$. The budget line has slope $= -\frac{p_1}{p_2}$. If $\frac{a}{b} \geq \frac{p_1}{p_2}$ so the indifference curves are at least as steep as the budget line, then it is optimal to choose $x_1 = \frac{m}{p_1}$ and $x_2 = 0$. This gives

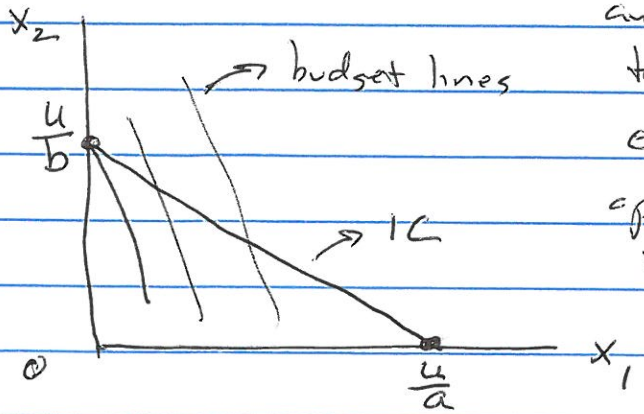
$$v(p,m) = \frac{am}{p_1}.$$



On the other hand if $\frac{a}{b} \leq \frac{p_1}{p_2}$ it is optimal to choose $x_1 = 0$ and $x_2 = \frac{m}{p_2}$. This gives $V(p,m) = \frac{bm}{p_2}$

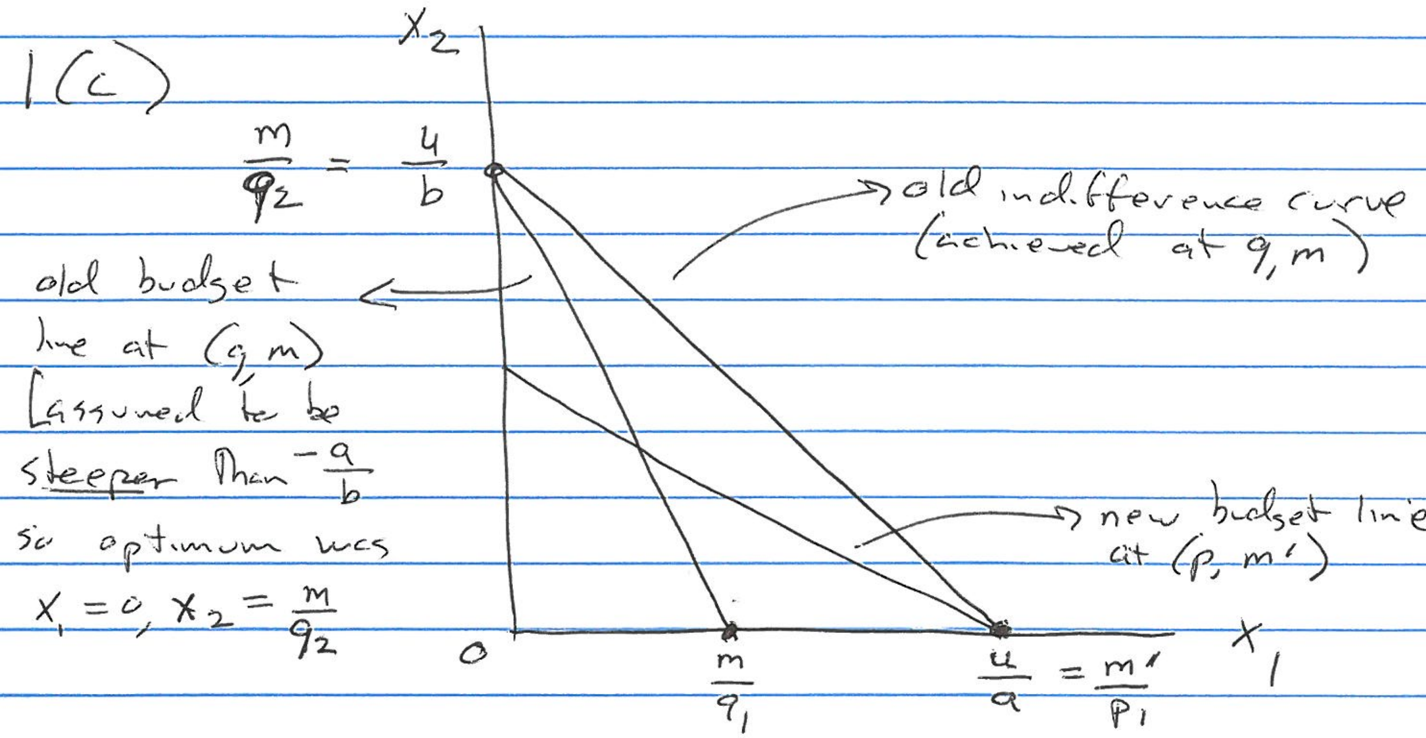
Combining these results, we have $V(p,m) = \max\left\{\frac{am}{p_1}, \frac{bm}{p_2}\right\}$.

(b) Fix an indifference curve by choosing some u. The IC has slope $= -\frac{a}{b}$. Then for given prices p, go to the lowest possible budget line. If $\frac{a}{b} \leq \frac{p_1}{p_2}$ so the budget lines are steeper than IC, it is optimal to choose $x_1 = 0$, $x_2 = \frac{u}{b}$. This gives $e(p,u) = \frac{p_2 u}{b}$. If $\frac{a}{b} \geq \frac{p_1}{p_2}$ it is optimal to choose $x_1 = \frac{u}{a}$, $x_2 = 0$. This gives $e(p,u) = \frac{p_1 u}{a}$. In general,



$$e(p,u) = \min\left\{\frac{p_1 u}{a}, \frac{p_2 u}{b}\right\}.$$

1(c)



[assuming new budget lines with p are flatter than $-\frac{a}{b}$ so now the optimum is a corner with $x_1 = \frac{m'}{p_1}$, $x_2 = 0$]
To achieve the same indifference curve as before m' must be chosen so that the bundle $x_1 = \frac{m'}{p_1}$, $x_2 = 0$ gives the same utility u as before.

$\Rightarrow \frac{m'}{p_1} = \frac{u}{a}$ where $\frac{m}{q_2} = \frac{u}{b} \Rightarrow u = \frac{mb}{q_2}$

$\Rightarrow m' = \frac{p_1 u}{a}$ $\qquad \Rightarrow \boxed{m' = \frac{u p_1}{a} = \frac{b}{a}\frac{p_1}{q_2} m}$

2(a) Suppose x^* does not solve $\min px$ subj to $u(x) \geq u^*$. Then there is some $x' \neq x^*$ such that $u(x') \geq u^*$ so x' is feasible, with $px' < px^*$ because x' achieves lower expenditure than x^*. By local non-satiation there is some x'' such that $px'' < px^*$ and $u(x'') > u(x')$. Also by non-satiation, $px^* = m$ because x^* maxes $u(x)$ s.t. $px \leq m$. Now we have $u(x'') > u(x') \geq u^* = u(x^*)$ and $px'' < m$. This contradicts the fact that x^* maxes $u(x)$ s.t. $px \leq m$.

2 (b) We want x^* to solve max $u(x)$ subject to $px = m$. For this to be true, x^* must satisfy the FOC for a max. Using a Lagrangean, the FOC are

$$\frac{\partial u(x^*)}{\partial x_i} = \lambda p_i \quad \text{for all } i = 1 \ldots n$$

and $$\sum_{i=1}^{n} p_i x_i^* = 1 \qquad \text{where we set } m^* = 1.$$

Multiply the first n equations by x_i^* to get

$$\frac{\partial u(x^*)}{\partial x_i} x_i^* = \lambda p_i x_i^* \qquad i = 1 \ldots n$$

$$\Rightarrow \quad \sum_i \frac{\partial u(x^*)}{\partial x_i} x_i^* = \lambda \sum_i p_i x_i^* = \lambda \qquad \text{using } m^* = 1.$$

So we set $$p_i = \frac{\frac{\partial u(x^*)}{\partial x_i}}{\sum_i \frac{\partial u(x^*)}{\partial x_i} x_i^*} \qquad \text{for all } i = 1 \ldots n.$$

Use these prices for p^* and use $m^* = 1$. For (p^*, m^*) the bundle satisfies the FOC for a max by construction. But the FOC is also sufficient for a max due to strict quasi-concavity.

2 (c) Given x^*, set $m = 1$ and let p be the prices p^* from part (b). Thus at $(p, 1)$ the consumer chooses x^*. Thus

$$V(p, 1) = u(x^*) = \max_x u(x) \text{ subject to } px = 1.$$

Consider any other prices p' such that $p'x^* = 1$. At these prices, x^* is feasible but not necessarily optimal, so

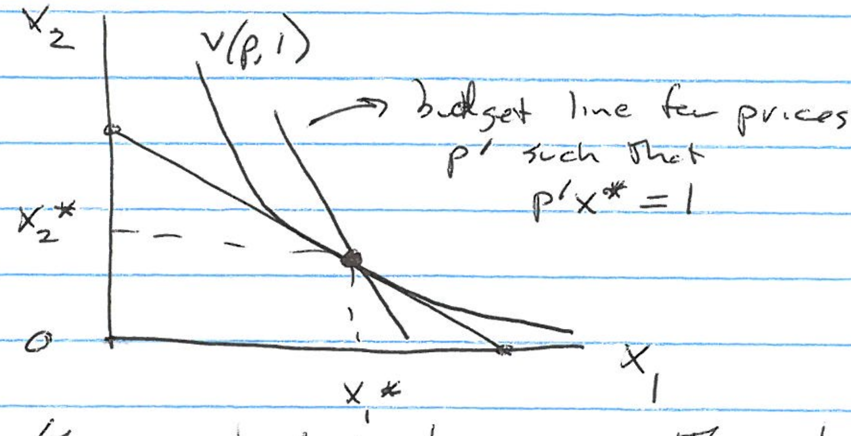
$$V(p', 1) \geq u(x^*) = V(p, 1). \quad \text{Therefore}$$

$$u(x^*) = V(p, 1) = \min_{p'} V(p', 1) \text{ subject to } p'x^* = 1.$$

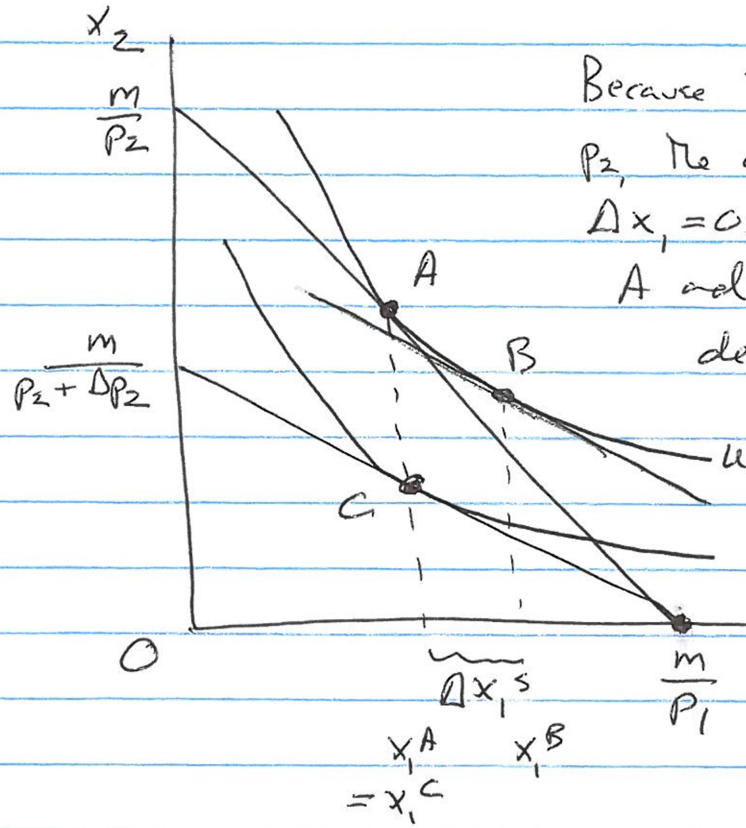
or simply $u(x^*) = \min_p v(p,1)$ subject to $px^* = 1$.

Intuition:

Suppose x^* is optimal for some initial budget line with prices p and $m=1$. This gives $v(p,1) = u(x^*)$. For any other $p' \neq p$ such that x^* is affordable (the new budget line passes through x^*) it is possible to move along the budget line and achieve a higher indifference curve, so $v(p',1) > v(p,1)$. Thus when we minimize $v(p,1)$ with respect to p subj to $px^*=1$, we get $u(x^*)$.



3 (a)



Because $x_1 = \frac{m}{2p_1}$ does not depend on p_2, the overall effect of $\Delta p_2 > 0$ is $\Delta x_1 = 0$. Thus the tangency points A and C must give the same demand $x_1^A = x_1^C$. We construct the substitution effect by going from point A to B.

The substitution effect Δx_1^S involves a movement along the old indifference curve u. The income effect $-\Delta x_1^S$ involves a parallel shift in the budget line at the new prices from B to C. This exactly cancels out the substitution effect.

3(b) Slutsky equation:

$$\begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial p_1} & \frac{\partial h_1}{\partial p_2} \\ \frac{\partial h_2}{\partial p_1} & \frac{\partial h_2}{\partial p_2} \end{bmatrix} - \begin{bmatrix} \frac{\partial x_1}{\partial m} x_1 & \frac{\partial x_1}{\partial m} x_2 \\ \frac{\partial x_2}{\partial m} x_1 & \frac{\partial x_2}{\partial m} \cdot x_2 \end{bmatrix}$$

so $$\frac{\partial h}{\partial p} = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial x_1}{\partial m} x_1 & \frac{\partial x_1}{\partial m} x_2 \\ \frac{\partial x_2}{\partial m} x_1 & \frac{\partial x_2}{\partial m} x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{m}{2p_1^2} & 0 \\ 0 & \frac{-m}{2p_2^2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2p_1}\frac{m}{2p_1} & \frac{1}{2p_1}\frac{m}{2p_2} \\ \frac{1}{2p_2}\frac{m}{2p_1} & \frac{1}{2p_2}\frac{m}{2p_2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{m}{4p_1^2} & \frac{m}{4p_1p_2} \\ \frac{m}{4p_1p_2} & -\frac{m}{4p_2^2} \end{bmatrix}$$

Yes, this is what we expect. The substitution matrix is symmetric and also negative semi-definite (note that the diagonal terms are strictly negative, so the Hicksian demands slope down, and the determinant of the overall matrix is zero, which is consistent with negative semi-definiteness)

3(c) $$X^1(p, m_1 \ldots m_n) = \sum_{i=1}^{n} x_i^1(p, m_i) = \sum_{i=1}^{n} \frac{m_i}{2p_1} = \frac{M}{2p_1}$$

where $M = \sum_i m_i$ and similarly

$$X^2(p, m_1 \ldots m_n) = \frac{M}{2p_2}$$

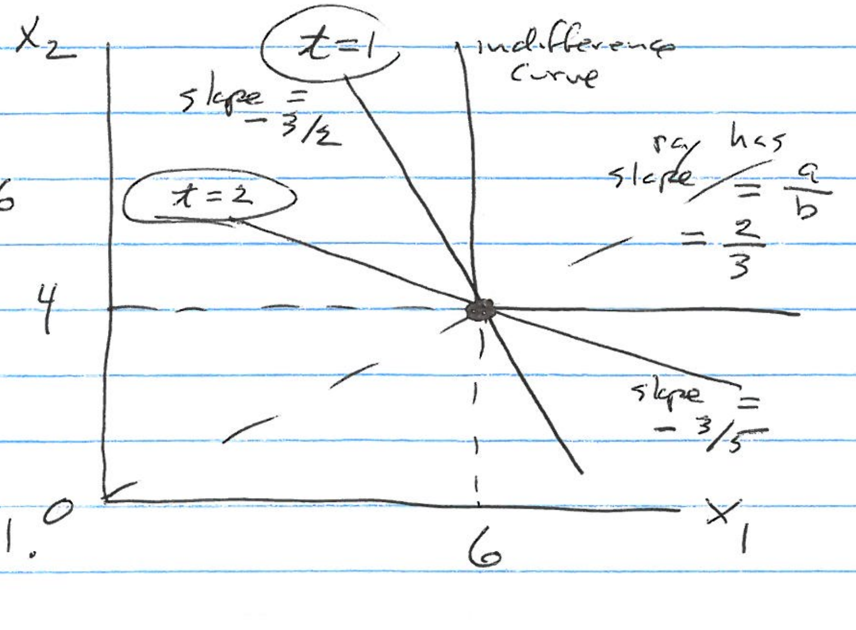
The aggregate demands only depend on total income M, not the distribution among individual consumers $(m_1 \ldots m_n)$. Because the aggregate demands X^1 and X^2 have the same form as the individual demands x_1 and x_2 (we just replace m by M), we would get the same results if we used the Slutsky equation to compute a substitution matrix for the aggregate demands. This strongly suggests that the indirect utility functions for individual consumers must satisfy the necessary and sufficient conditions for aggregation, ie they have the Gorman form $v(p, m_i) = a_i(p) + b(p) m_i$.
[You didn't have to prove this, but in fact this is true; the Marshallian demands come from the direct utility function $u(x) = x_1^{1/2} x_2^{1/2}$ which is linearly homogeneous and therefore homothetic. Because all consumers have the same preferences, this satisfies the Gorman conditions]

4 (a) Yes, the Leontief utility function $u(x) = \min\{a x_1, b x_2\}$ could generate these observations.

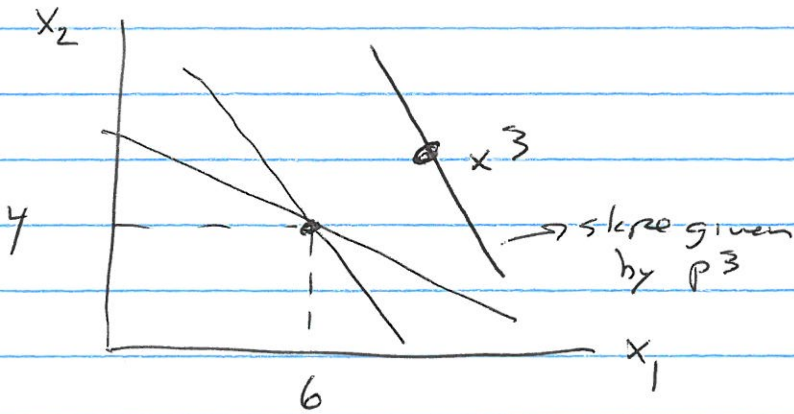
In period $t=1$ the consumer can afford $(6,4)$ and has income $m^1 = (6)(3) + (4)(2) = 26$
This point is optimal.

In period $t=2$, the consumer can afford $(6,4)$ and has income $m^2 = (6)(3) + (4)(5) = 38$

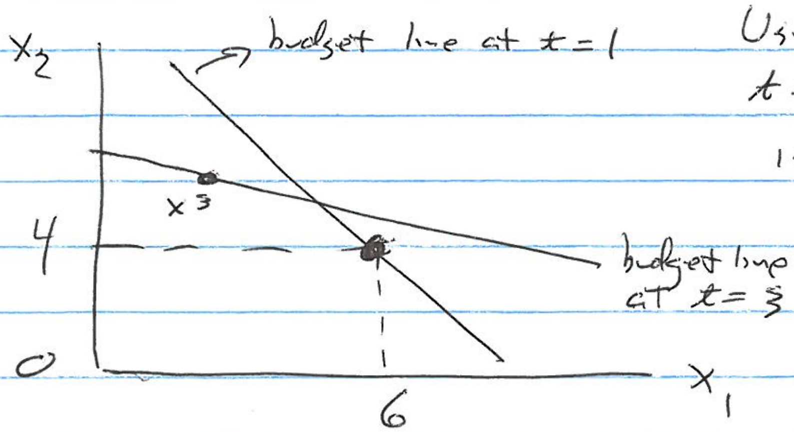
Again the point $(6,4)$ is optimal.



4(b) GARP says that (6, 4) is at least as good as any point on or below the budget lines in periods $t = 1, 2$. However, it says nothing about how (6, 4) is ranked relative to some point x^3 located <u>above</u> both of the previous budget lines. So choose any such point for x^3 and construct a budget line through x^3 using p^3 (and income $m^3 = p^3 x^3$)



4(c) GARP says (6, 4) is at least as good as anything that was feasible in period $t = 1$ (ignore $t = 2$ for simplicity). Moreover, we have non satiation, so (6, 4) is strictly preferred to anything below the budget line from $t = 1$. Therefore if x^3 is <u>below</u> the period $t = 1$ budget line but at the prices p^3 the point (6, 4) is also available, this would be a contradiction of GARP.

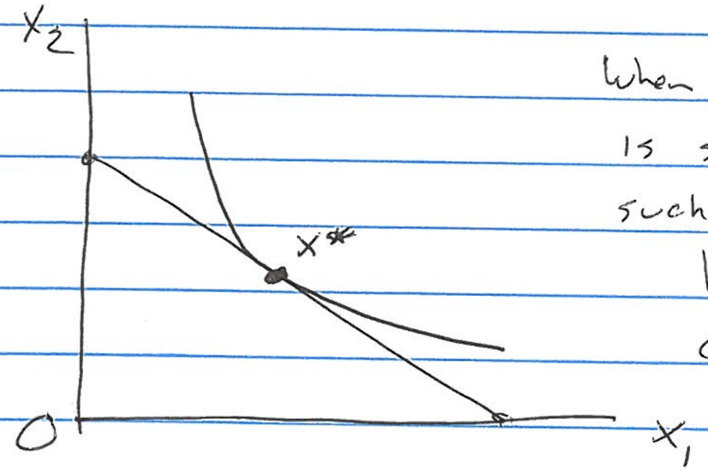


Using the graph: period $t = 1$ revealed that (6, 4) is strictly preferred to x^3, but period $t = 3$ revealed that x^3 is strictly preferred to (6, 4).

5(a) Sufficient SOC:

$$h \frac{\partial^2 u(x^*)}{\partial x^2} h < 0 \text{ for all } h \neq 0 \text{ such that } ph = 0$$

where x^* satisfies FOC.



When the sufficient SOC holds, there is some neighborhood of x^* such that any small movement $h = x - x^*$ along the budget constraint must reduce utility.

The SOC comes from a Taylor series expansion of $u(x)$ around x^*. We want $u(x) - u(x^*) < 0$ for all feasible x in a neighborhood of x^*. The first order terms of the expansion drop out because $\frac{\partial u(x^*)}{\partial x} = \lambda p$ from the FOC, and for x to be a feasible deviation we need $px = px^* = m \Rightarrow ph = 0$. For small $h \neq 0$ we can ignore higher order terms as long as a strict inequality holds for the second order terms.

5(b) Write $q = t q_0$ for some fixed price vector q_0.

Use the indirect utility function

$$v(p, t, m) = \max u(x, z) \text{ subject to } px + t q_0 z = m$$

Define the composite commodity $Z = q_0 z$ and derive a direct utility function for (x, Z) in the usual way:

$$w(x, Z) = \min_{(p, t)} v(p, t, 1) \text{ subject to } px + tZ = 1.$$

The Marshallian demands corresponding to $u(x, Z)$ can be written as $x(p, t, m)$ and $Z(p, t, m)$. Using the zero degree homogeneity of the Marshallian demands, $x(p, t, m) = x\left(\frac{p}{t}, 1, \frac{m}{t}\right)$

So the demands for the individual x goods only depend on the price vector P relative to the price index t, and income m relative to the index t.

5(c) The student is thinking of leisure as an ordinary good whose price is equal to the wage. If income did not depend on the wage, the student would be right: If the Marshallian demand for some good does not change in response to the good's own price, it would have to be "almost" Giffen (ie an inferior good with an income effect strong enough to cancel out the substitution effect). But the student is overlooking the fact that the wage is not only the price of leisure — it also helps to determine income. Once we take this into account leisure could be a normal good. If someone is a net supplier of labor time, and the wage goes up, they get more income and therefore demand more leisure. This could offset the usual substitution effect and give a zero net response of leisure to changes in the wage rate.